Pearson Edexcel Level 3

GCE Further Mathematics

Advanced Subsidiary

Further Pure Mathematics 2

Specimen paper Time: 50 minutes	Paper Reference(s) 8FM0/22
You must have: **Mathematical Formulae and Statistical Tables, calculator**	

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use black ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need*.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 4 questions in this section of the paper. The total mark is 40.
- The marks for each question are shown in brackets – *use this as a guide as to how much time to spend on each question*.

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Answer ALL questions. Write your answers in the spaces provided.

2.1: The Axioms For A Group | 2.2: Cayley Tables & Finite Groups | 2.3: Order & Subgroups

1. (i) The set $G = \{1, 3, 4, 9, 10, 12\}$ forms a group under the operation of multiplication modulo 13.

$a * b = a \times b \pmod{13}$

(a) Copy and complete the Cayley table below, Table 1, for this group.

(3)

(b) Find a subgroup of $(G, \times_{13})$ of order 3.

(1)

(c) Explain why there can be no subgroup of $(G, \times_{13})$ of order 4.

(1)

(ii) Determine whether the set $\{1, 3, 6, 9, 12\}$ under the operation of multiplication modulo 15 forms a group.

(3)

$\times_{13}$	**1**	**3**	**4**	**9**	**10**	**12**
1	1	3	4	9	10	12
3	3	9	12	1	4	10
4	4	12	3	10	1	9
9	9	1	10	3	12	4
10	10	4	1	12	9	3
12	12	10	9	4	3	1

Table 1

(Total for Question 1 is 8 marks)

b.

$1^1 = 1$	$3^1 = 3$	$4^1 = 4$	$9^1 = 9$	$10^1 = 10$	$12^1 = 12$
$1^2 = 1$	$3^2 = 9$	$4^2 = 3$	$9^2 = 3$	$10^2 = 9$	$12^2 = 1$
$1^3 = 1$	$3^3 = 1$	$4^3 = 12$	$9^3 = 1$	$10^3 = 12$	$12^3 = 12$
$1^4 = 1$	$3^4 = 3$	$4^4 = 9$	$9^4 = 9$	$10^4 = 3$	$12^4 = 1$
$1^5 = 1$	$3^5 = 9$	$4^5 = 10$	$9^5 = 3$	$10^5 = 4$	$12^5 = 12$
$1^6 = 1$	$3^6 = 1$	$4^6 = 1$	$9^6 = 1$	$10^6 = 1$	$12^6 = 1$
↓ order 1	↓ order 3	↓ order 6	↓ order 3	↓ order 6	↓ order 2

$\{1, 3, 9\}$ //

c. The order of the group is 6, and 4 does not divide 6, so there can be no subgroup of order 4 by Lagrange's theorem. //

d.

$\times_{15}$	1	3	6	9	12
1	1	3	6	9	12
3	3	9	3	12	6
6	6	3	6	9	12
9	9	12	9	6	3
12	12	6	12	3	9

← No element 1 in 2^{nd} row/column ∴ 3 has no inverse

Since 1 is not in every row, not every row has an inverse

↳ Since no inverse, cannot be a group. //

5.2: Reducing Matrices to Diagonal Form

2. The two sequences x_n and y_n satisfy the recurrence relations

$$x_1 = 2 \qquad x_{n+1} = 5x_n - y_n$$
$$y_1 = 1 \qquad y_{n+1} = 3x_n + y_n \qquad n \geq 1.$$

(a) Find the value of x_3.

(2)

These recurrence relations can be written in matrix form as

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{A}\begin{pmatrix} x_n \\ y_n \end{pmatrix}, \qquad n \geq 1,$$

where $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$.

(b) Find

(i) the eigenvalues of **A**,

(ii) the corresponding eigenvectors of **A**.

(5)

(c) Hence write down matrices **P** and **D** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where **D** is a diagonal matrix.

(2)

New sequences u_n and v_n can be formed from x_n and y_n using the transformation

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

(d) Show that $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{D}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$.

(2)

(e) Hence find closed form expressions for the original sequences x_n and y_n.

(5) ~~**(2)**~~

(Total for Question 2 is 16 marks)

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a. $x_2 = 5x_1 - y_1$

$x_2 = 5(2) - (1) = 9$

$y_2 = 3x_1 + y_1$

$y_2 = 3(2) + (1) = 7$

find x_3 manually but subbing in known values into recurrence eq^n

$x_2 = 9$, $y_2 = 7$

$x_3 = 5x_2 - y_2$

$x_3 = 5(9) - (7) = 38$

$x_3 = 38$ //

bi. characteristic eq^n: $\det(A - \lambda I) = 0$

$A - \lambda I = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$= \begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix}$

$\det\left[\begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix}\right] = 0$

$\det(M)$ where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det(M) = (a)(d) - (b)(c)$

$(5-\lambda)(1-\lambda) - (3)(-1) = 0$

$(\lambda^2 - 6\lambda + 5) - (-3) = 0$

$\lambda^2 - 6\lambda + 5 + 3 = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$(\lambda - 4)(\lambda - 2) = 0$

$\lambda = 4$ or $\lambda = 2$ ← eigenvalues of A

eigenvalues of A: 4 or 2

ii. $A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 5x - y \\ 3x + y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$

$5x - y = 4x$ ①
$3x + y = 4y$ ②

① $5x - y = 4x$
$x = y$

when $x = 1$, $y = 1$
$\therefore$ eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for eigenvalue 4.

$A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 5x - y \\ 3x + y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

$5x - y = 2x$ ①
$3x + y = 2y$ ②

① $5x - y = 2x$
$3x = y$

when $x = 1$, $y = 3$
$\therefore$ eigenvector is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ for eigenvalue 2.

c. $D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ or $D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ ← eigenvalues must be diagonal matrix

corresponding P n D matrices

$P = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ or $P = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$

$D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ OR $D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ // Either answer is correct

d. since $\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$, $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = P^{-1}\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$

sub in (n+1) into n

$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A\begin{pmatrix} x_n \\ y_n \end{pmatrix}$

↳ sub this into P^{-1} matrix formed.

$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = P^{-1}A\begin{pmatrix} x_n \\ y_n \end{pmatrix}$

$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$

multiply P on both sides.

This creates $PP^{-1} = I$

∴ isolating $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$

$P\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \underbrace{PP^{-1}}_{I}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$

$P\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$

↳ sub into $P^{-1}A\begin{pmatrix} x_n \\ y_n \end{pmatrix}$

$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \underbrace{P^{-1}AP}_{D}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$

as stated in part (c)

$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = D\begin{pmatrix} u_n \\ v_n \end{pmatrix}$ //

e. we will use $D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ however can also use $D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$

sub in

$$\begin{pmatrix} U_{n+1} \\ V_{n+1} \end{pmatrix} = D \begin{pmatrix} U_n \\ V_n \end{pmatrix}$$

$$\begin{pmatrix} U_{n+1} \\ V_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} U_n \\ V_n \end{pmatrix}$$

$$\begin{pmatrix} U_{n+1} \\ V_{n+1} \end{pmatrix} = \begin{pmatrix} 2U_n \\ 4V_n \end{pmatrix}$$

$U_{n+1} = 2U_n$

$V_{n+1} = 4V_n$

can be rewritten as:

$U_n = U_1 \times 2^{n-1}$

$V_n = V_1 \times 4^{n-1}$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = P \begin{pmatrix} U_n \\ V_n \end{pmatrix} \Rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} U_n \\ V_n \end{pmatrix}$$

use corresponding P matrix and sub in

$$\Rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} U_n + V_n \\ 3U_n + V_n \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} U_1 \times 2^{n-1} + V_1 \times 4^{n-1} \\ 3(U_1 \times 2^{n-1}) + V_1 \times 4^{n-1} \end{pmatrix}$$

using $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ sub in $n=1$ and solve for U_1 and V_1.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} U_1 \times 2^{1-1} + V_1 \times 4^{1-1} \\ 3(U_1 \times 2^{1-1}) + V_1 \times 4^{1-1} \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} U_1 + V_1 \\ 3U_1 + V_1 \end{pmatrix}$$

Solve simultaneously to solve for u_1 and v_1.

$u_1 + v_1 = 2$

$3u_1 + v_1 = 1$ $\ominus$

$-2u_1 = 1$

$u_1 = -\frac{1}{2}$

sub in worked out value of u_1 to find v_1

$v_1 = 2 - u_1$

$v_1 = 2 - \left(-\frac{1}{2}\right)$

$v_1 = \frac{5}{2}$

$u_1 = -\frac{1}{2}$ and $v_1 = \frac{5}{2}$

↳ sub back into x_n and y_n eq^n.

$x_n = \left(-\frac{1}{2}\right)2^{n-1} + \left(\frac{5}{2}\right)4^{n-1}$

$y_n = \left(-\frac{3}{2}\right)2^{n-1} + \left(\frac{5}{2}\right)4^{n-1}$ //

3.1: Loci on an Argand Diagram

3. (a) Sketch, on an Argand diagram, the arc given by the locus of points z in the complex plane satisfying

$$\arg\left(\frac{z}{z-25\mathrm{i}}\right)=\frac{11\pi}{12}.$$

← LOCUS EXPLAINED @ END OF Q

(2)

The centre of the circle containing this arc is $x + y\mathrm{i}$ where

$$x=-\frac{25}{2}\left(2+\sqrt{3}\right).$$

(b) Find the radius of this circle, giving your answer to 3 decimal places.

(4)

In a game, players take turns to roll balls along a horizontal surface from a fixed starting position to a fixed target point 25 metres away.

The game can be modelled in the complex plane, with the starting position at the origin and the target at the point 25i, where the units are metres.

In a particular game, the first player's ball has come to rest with its centre at 0.5 + 23.25i. The second player's ball is following the path sketched in part (a).

Given that the radius of each ball is approximately 5 centimetres,

(c) use this model to determine whether the second player's ball will hit the first player's ball.

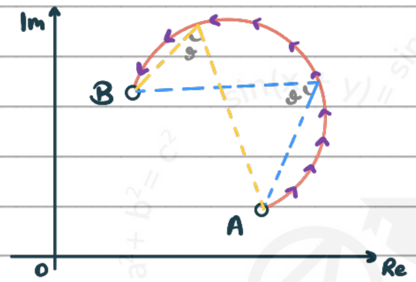
(4)

(Total for Question 4 is 10 marks)

The locus of points z that satisfy $\arg\left(\frac{z-a}{z-b}\right) = \theta$

where $\theta \in \mathbb{R}$, $\theta > 0$ and $a, b \in \mathbb{C}$, is an arc of a circle with endpoints A and B representing the complex no.s a and b, respectively.

The locus is the arc of a circle drawn anticlockwise from A to B.



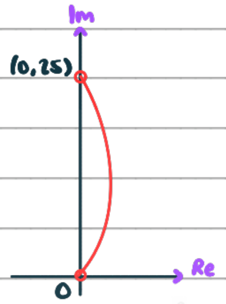
$\arg\left(\frac{z-a}{z-b}\right) = \theta$

if $\theta < \frac{\pi}{2}$ then the locus is a major arc of a circle
if $\theta = \frac{\pi}{2}$ then the locus is a semi-circle.
if $\theta > \frac{\pi}{2}$ then the locus is a minor arc of a circle

3a. $\arg\left(\dfrac{z}{z-25i}\right) = \dfrac{11\pi}{12}$

a : $0 + 0i$ $\quad(0,0)$

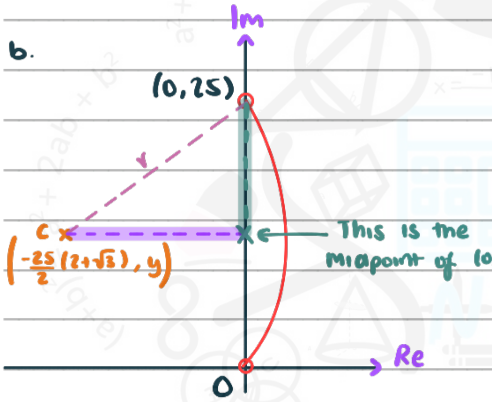
b : $0 + 25i$ $\quad(0,25)$



→ as $\frac{11\pi}{12} > \frac{\pi}{2}$, the locus is a minor arc of the circle

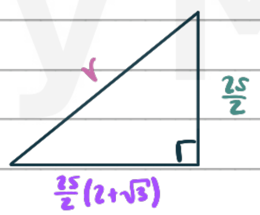
→ arc drawn anticlockwise from O to 25i.

b.



This length is $\frac{25}{2}(2+\sqrt{3})$

This length is $\frac{25}{2}$

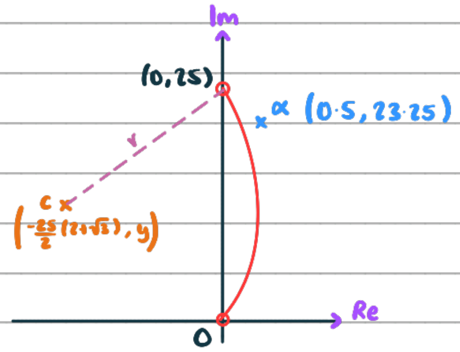


$r = \sqrt{\left[\frac{25}{2}(2+\sqrt{3})\right]^2 + \left[\frac{25}{2}\right]^2}$ pythagoras theorem to find r

$r = 48.29629131$

$r \approx 48.3$ (3s.f.)

c



Firstly check distance from centre (c) to 1st ball (α)
Then check difference in distance between |cα| and radius (r) to see if balls collided.

Firstly, work out circle centre y-coord: This is midpoint of (0,0) and (0,25)

$$y: \left(\frac{0+25}{2}\right) = \frac{25}{2}$$

Centre: $\left(-\frac{25}{2}(2+\sqrt{3}), \frac{25}{2}\right)$

$$|c\alpha| = \sqrt{\left[-\frac{25}{2}(2+\sqrt{3}) - 0.5\right]^2 + \left[\frac{25}{2} - 23.25\right]^2}$$

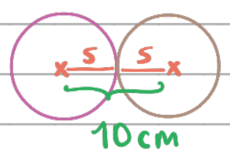
$|c\alpha| = 48.36057164$

Difference in distance between 2 balls:

$(48.36057164) - (48.29629131) = 0.06428033\text{m}$

$= 6.428033\text{cm}.$

For balls to have collided, difference $\leq$ 10 cm



any less than 10 and balls have gone closer inwards and hence collided

$6.428\ldots \text{ cm} < 10\text{cm}$

∴ Balls will collide //

1.4 : Divisibility Tests

4. A positive integer n satisfies the inequality $99 < n < 1000$.

The integer n is written as a three digit number abc such that

$$n = 100a + 10b + c,$$

where the digits a, b, and c are each integers between 0 and 9 inclusive, $a \neq 0$.

$1 \leq a \leq 9,\ 0 \leq b, c \leq 9$

The integer n has the following three properties:

- n is divisible by 11, ← fact # 1
- the sum of the digits of n is odd, ← fact # 2
- $n \equiv 5 \pmod 9$. ← fact # 3

Use this information to find all possible values for n, making your reasoning clear.

(Total for Question 4 is 6 marks)

TOTAL FOR PAPER IS 40 MARKS

4.

#1: $a-b+c=11k$

max. value of $a-b+c$ is $(9)-(0)+(9)=18$

$\therefore$ max. value of $a-b+c=18$

$\therefore$ k can only either be 0 or 1

$\hookrightarrow a-b+c=0$ or 11

#2 : $a+b+c=2q+1 \quad q\in\mathbb{Z}$

#3: $100a+10b+c\equiv 5 \pmod 9$

$100\equiv 1 \bmod 9$

$10 \equiv 1 \bmod 9$

can now be rewritten as

$a+b+c\equiv 5 \bmod 9$

* possible values of $a+b+c$: 5, 14, 23, 32, ...

Using facts ① and ②, $k=1$.

This is because if $a-b+c=0$, then $a+b+c=2b$.

$2b$ is always even — this contradicts #2.

$\therefore a-b+c=11$

From facts ② and ③, $\underset{\substack{\text{min value}\\(1)+(0)+(0)}}{1} \leq a+b+c \leq \underset{\substack{\text{max value}\\(9)+(9)+(9)}}{27}$

$\therefore$ $a+b+c$ can only either be 5 or 23.

$a+b+c\neq 14$ as #2 states $a+b+c$ = odd no.

$\rightarrow a+b+c=5$

$\rightarrow a+b+c=23$

$a-b+c=11 \quad (1)$

$a+b+c=5 \quad (2)$

$a+b+c=23 \quad (3)$

Solve simultaneous eqⁿs between (1) and (2) AND (1) and (3).

(1) and (2)

$a - b + c = 11$

$a + b + c = 5$ ⊝

$-2b = 6$

$b = -3$

solving for b

(1) and (3)

$a - b + c = 11$

$a + b + c = 23$ ⊝

$-2b = -12$

$b = 6$

either $b = -3$ or $b = 6$

since $0 \leq b \leq 9$, $b = 6$

$a + b + c = 23$

$a + 6 + c = 23$

$a + c = 17$

$1 \leq a \leq 9$ and $0 \leq c \leq 9$

$a = 8$ or 9

$c = 9$ or 8

only 2 combinations that will work

$n = 869$ or 968 //